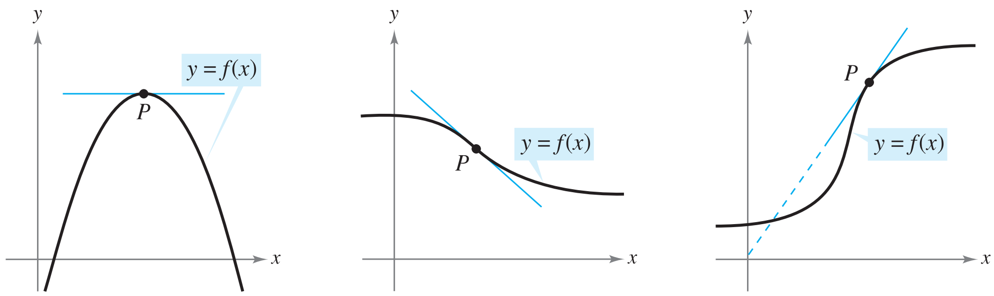
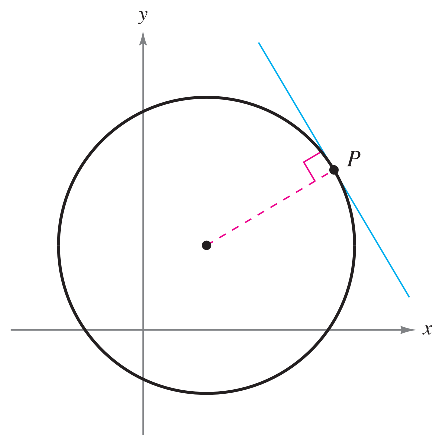
Math 150, *Lecture Notes- Bonds* **Name____________**

Section 2.1 *The Derivative and the Tangent Line Problem*

Can we say that a tangent line to a curve is a line that touches the curve at one point and does not pass through the curve at any other point?

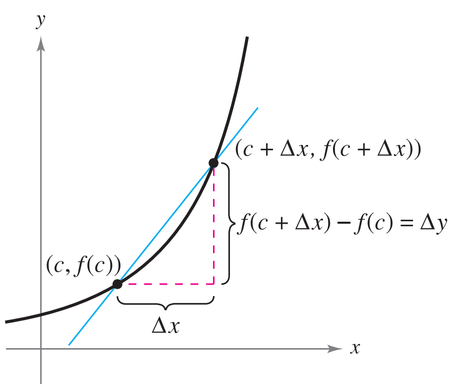


Tangent line to a curve at a point



Tangent line to a circle

Eventually, we see that the problem of finding a tangent line at a point *P*, becomes an issue of finding the slope of a the tangent line at a point *P*. We consider this problem by using a secant line through the point of tangency, $\left(c, f\left(c\right)\right)$, and a second point on the curve $\left(c+\Delta x, f\left(c+\Delta x\right)\right)$.



The secant line through $(c, f(c))$ and $(c + \Delta x, f(c + \Delta x))$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m_{\text{sec}} = \frac{f(c + \Delta x) - f(c)}{(c + \Delta x) - c} \qquad \frac{\text{Change in } y}{\text{Change in } x}$$

$$\boxed{m_{\text{sec}} = \frac{f(c + \Delta x) - f(c)}{\Delta x}.} \qquad \text{Slope of secant line}$$

The slope of the secant line through these two points is given by the following formula called the difference quotient, where Δx is called *the change in x*. In corresponding fashion, the numerator is called *the change in y*, $\Delta y = f\left(c+\Delta x\right) - f\left(c\right)$.

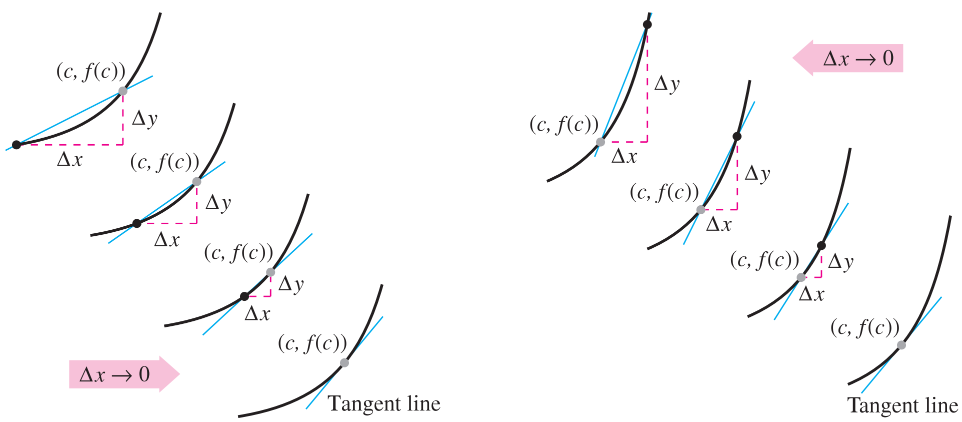
Definition of Tangent Line with Slope m

If f is defined on an open interval containing c, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = m_{tan}$$

exists, then the line passing through $(c, f(c))$ with slope m is the **tangent line** to the graph of f at the point $(c, f(c))$.

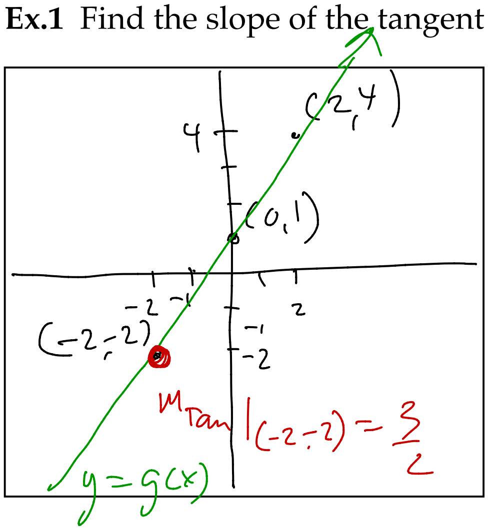
The slope of the tangent line to the graph of f at the point $(c, f(c))$ is also called the **slope of the graph of f at $x = c$.**



Tangent line approximations

$y = mx + b$

Ex.1 Find the slope of the tangent line to the graph of $g(x) = \frac{3}{2}x + 1$ at the point $(-2, -2)$. $(c, g(c))$



$$\lim_{\Delta x \to 0} \frac{g(-2+\Delta x) - g(-2)}{\Delta x} = m_{tan}\Big|_{(-2,-2)}$$

$$\lim_{\Delta x \to 0} \frac{\left[\frac{3}{2}(-2+\Delta x) + 1\right] - (-2)}{\Delta x} =$$

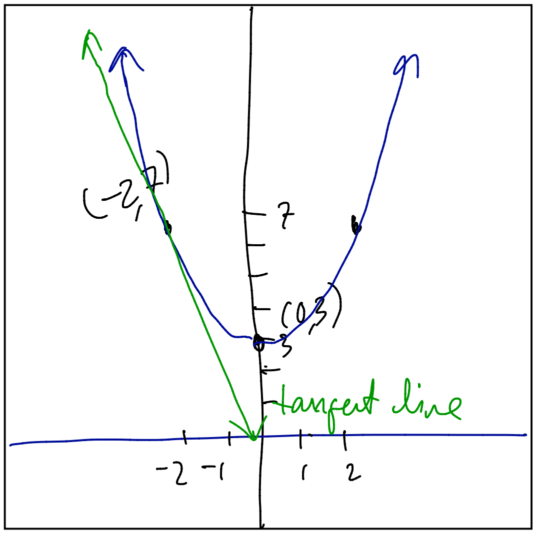
$$\lim_{\Delta x \to 0} \frac{-3 + \frac{3}{2}\Delta x + 1 + 2}{\Delta x} =$$

$$\lim_{\Delta x \to 0} \frac{\frac{3}{2} \cdot \cancel{\Delta x}}{\cancel{\Delta x}} = \lim_{\Delta x \to 0} \frac{3}{2} = \frac{3}{2} = m_{tan}\Big|_{(-2,-2)}$$

FOIL: $(-2+\Delta t)(-2+\Delta t)$

$= 4 - 2\Delta t - 2\Delta t + \Delta t^2$

Ex.2 Find the slope of the tangent line to the graph of $h(t) = t^2 + 3$ at the point $(-2, 7)$.



$$m_{tan}\Big|_{(-2,7)} = \lim_{\Delta t \to 0} \frac{h(-2+\Delta t) - h(-2)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{[(-2+\Delta t)^2 + 3] - (7)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{4 - 4\Delta t + \Delta t^2 + 3 - 7}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\cancel{\Delta t}(-4 + \Delta t)}{\cancel{\Delta t}}$$

$$= \lim_{\Delta t \to 0} (-4 + \Delta t)$$

$$= -4 + 0$$

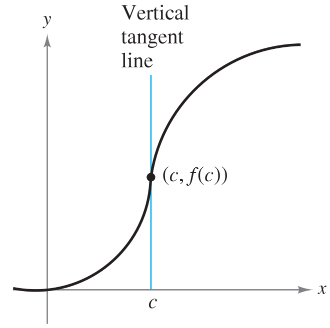
$$m_{tan}\Big|_{(-2,7)} = -4$$

$m_{Tan}\Big|_{(-2,7)} < 0$

The definition of a tangent line to a curve does not cover the possibility of a vertical tangent line. For vertical tangent lines, you can use the following definition. If f is continuous at c and

$$\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \infty \quad \text{or} \quad \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = -\infty$$

the vertical line $x = c$ passing through $(c, f(c))$ is a **vertical tangent line** to the graph of f. For example, the function shown in Figure 2.7 has a vertical tangent line at $(c, f(c))$. If the domain of f is the closed interval $[a, b]$, you can extend the definition of a vertical tangent line to include the endpoints by considering continuity and limits from the right (for $x = a$) and from the left (for $x = b$).



The graph of f has a vertical tangent line at $(c, f(c))$.
Figure 2.7

Definition of the Derivative of a Function

The **derivative** of f at x is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

provided the limit exists. For all x for which this limit exists, f' is a function of x.

Be sure you see that the derivative of a function of x is also a function of x. This "new" function gives the slope of the tangent line to the graph of f at the point $(x, f(x))$, provided that the graph has a tangent line at this point.

The process of finding the derivative of a function is called **differentiation.** A function is **differentiable** at x if its derivative exists at x and is **differentiable on an open interval (a, b)** if it is differentiable at every point in the interval.

In addition to $f'(x)$, which is read as "f prime of x," other notations are used to denote the derivative of $y = f(x)$. The most common are

$$f'(x), \quad \frac{dy}{dx}, \quad y', \quad \frac{d}{dx}[f(x)], \quad D_x[y].$$

Notation for derivatives

The notation dy/dx is read as "the derivative of y *with respect to* x" or simply "dy, dx." Using limit notation, you can write

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= f'(x). \end{aligned}$$

$f'(x) = 3x^2 + 2x$

1
1 1
1 2 1
1 3 3 1

Ex.3 Find the derivative of $f(x) = x^3 + x^2$ by the limit process.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{[(x+\Delta x)^3 + (x+\Delta x)^2] - [x^3 + x^2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 + x^2 + 2x\Delta x + \Delta x^2 - x^3 - x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 + 2x\Delta x + \Delta x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}[3x^2 + 3x\Delta x + \Delta x^2 + 2x + \Delta x]}{\cancel{\Delta x}}$$

$$= \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + \Delta x^2 + 2x + \Delta x)$$

$$= 3x^2 + 3x \cdot 0 + 0^2 + 2x + 0$$

$$f'(x) = 3x^2 + 2x$$

Ex.4 Find the derivative of $g(x) = \frac{1}{x^2}$ by the limit process.

$$g'(x) = \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{1}{(x+\Delta x)^2} - \frac{1}{x^2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{\frac{1}{(x+\Delta x)^2} - \frac{1}{x^2}}{\frac{\Delta x}{1}}\right] \cdot \left[\frac{\frac{x^2(x+\Delta x)^2}{1}}{\frac{x^2(x+\Delta x)^2}{1}}\right]$$

$$= \lim_{\Delta x \to 0} \frac{x^2 - (x+\Delta x)^2}{\Delta x \cdot x^2 \cdot (x+\Delta x)^2}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 - [x^2 + 2x\Delta x + \Delta x^2]}{\Delta x \cdot x^2 \cdot (x+\Delta x)^2}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 - x^2 - 2x\Delta x - \Delta x^2}{\Delta x \cdot x^2 \cdot (x+\Delta x)^2}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x} \cdot (-2x - \Delta x)}{\cancel{\Delta x} \cdot x^2 \cdot (x+\Delta x)^2}$$

$$= \lim_{\Delta x \to 0} \frac{-2x - \Delta x}{x^2 \cdot (x+\Delta x)^2}$$

$$= \frac{\lim_{\Delta x \to 0} (-2x - \Delta x)}{\lim_{\Delta x \to 0} x^2 (x+\Delta x)^2}$$

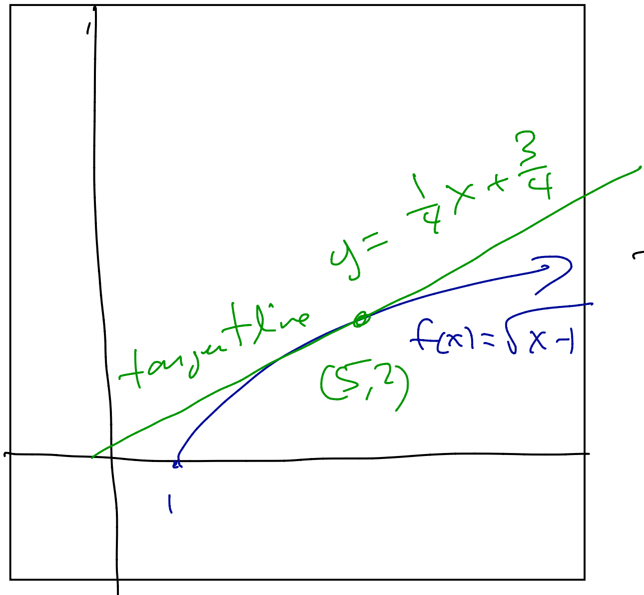
$$= \frac{-2x - 0}{x^2 (x+0)^2}$$

$$g'(x) = \frac{-2\cancel{x}}{x^2 \cdot x^{\cancel{2}}}$$

$$g'(x) = \frac{-2}{x^3}$$

$$y - y_1 = m_{tan}(x - x_1) \qquad (5,2) = (x_1, y_1)$$

Ex.5 Find the derivative of $f(x) = \sqrt{x-1}$ by the limit process. Use this information to find an equation of the tangent line to the graph at the point $(5,2)$. Use a graphing utility to graph $f(x) = \sqrt{x-1}$ and it's tangent line at this point. Use the *derivative* feature of this graphing utility to confirm your results.



$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{(x+\Delta x)-1} - \sqrt{x-1}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left[\frac{\sqrt{x+\Delta x-1} - \sqrt{x-1}}{\Delta x}\right] \cdot \left[\frac{\sqrt{x+\Delta x-1} + \sqrt{x-1}}{\sqrt{x+\Delta x-1} + \sqrt{x-1}}\right]$$

Difference of Squares

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x-1) - (x-1)}{\Delta x \cdot [\sqrt{x+\Delta x-1} + \sqrt{x-1}]}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x} \cdot 1}{\cancel{\Delta x} \cdot [\sqrt{x+\Delta x-1} + \sqrt{x-1}]}$$

$$= \lim_{\Delta x \to 0} \frac{1}{[\sqrt{x+\Delta x-1} + \sqrt{x-1}]}$$

$$= \frac{1}{\sqrt{x+0-1} + \sqrt{x-1}}$$

$$= \frac{1}{\sqrt{x-1} + \sqrt{x-1}}$$

$$\boxed{f'(x) = \frac{1}{2\sqrt{x-1}}}$$

$$f'(5) = m_{tan}\Big|_{(5,2)}$$

$$f'(5) = \frac{1}{2\sqrt{(5)-1}}$$

$$f'(5) = \frac{1}{2\sqrt{4}}$$

$$f'(5) = \frac{1}{2 \cdot 2}$$

$$f'(5) = \frac{1}{4}$$

$$y - y_1 = m_{tan} \cdot (x - x_1)$$

$$y - (2) = \frac{1}{4}(x - 5)$$

$$y - 2 = \frac{1}{4}x - \frac{5}{4}$$

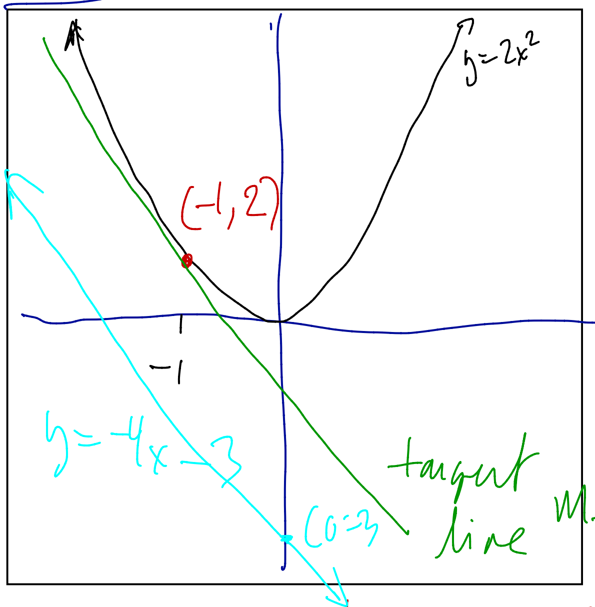
$$y - 2 + 2 = \frac{1}{4}x - \frac{5}{4} + \frac{8}{4}$$

$$\boxed{y = \frac{1}{4}x + \frac{3}{4}}$$

tangent line equation

SAME slope $\quad y - y_1 = m_{Tan}(x - x_1)$

Ex.6 Find the derivative of $f(x) = 2x^2$ by the limit process. Find an equation of the line that is tangent to graph of $f(x) = 2x^2$ and parallel to $4x + y + 3 = 0$.



$m_{Tan} = -4$

$4x + y + 3 = 0$

$y = -4x - 3$

$m = -4$

$f(-1) = 2(-1)^2$

$f(-1) = 2 \cdot 1$

$f(-1) = 2$

$m_{Tan} = -4$

$m_{TAN} = f'(x)$

$m_{Tan} = 4x$

Solve: $-4 = 4x$

$x = -1$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{2(x + \Delta x)^2 - 2x^2}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{2[x^2 + 2x\Delta x + \Delta x^2] - 2x^2}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{2x^2 + 4x\Delta x + 2\Delta x^2 - 2x^2}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{4x\Delta x + 2\Delta x^2}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta x[4x - 2\Delta x]}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} (4x - 2\Delta x)$$

$$f'(x) = 4x - 2 \cdot 0$$

$$\boxed{f'(x) = 4x}$$

$$y - y_1 = m_{Tan}(x - x_1)$$

$$y - (2) = -4(x - (-1))$$

$$y - 2 = -4x - 4$$

$$y = -4x - 2$$

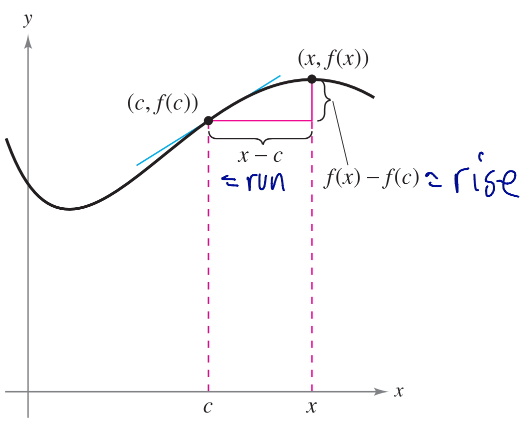
The following alternative limit form of the derivative is useful in investigating the relationship between differentiability and continuity. The derivative of f at c is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \qquad \text{Alternative form of derivative}$$

provided this limit exists (see Figure 2.10). (A proof of the equivalence of this form is given in Appendix A.) Note that the existence of the limit in this alternative form requires that the one-sided limits

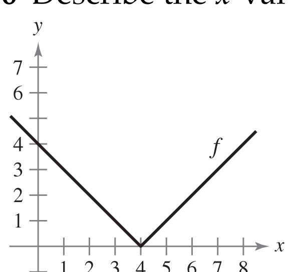
$$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \quad \text{and} \quad \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

exist and are equal. These one-sided limits are called the **derivatives from the left and from the right,** respectively. It follows that f is **differentiable on the closed interval $[a, b]$** if it is differentiable on (a, b) and if the derivative from the right at a and the derivative from the left at b both exist.

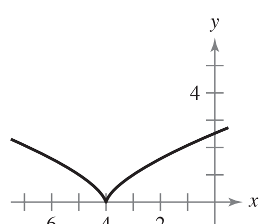


As x approaches c, the secant line approaches the tangent line.
Figure 2.10

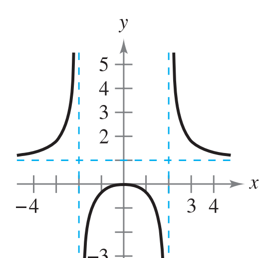
Ex.6 Describe the x-values at which the functions are differentiable:



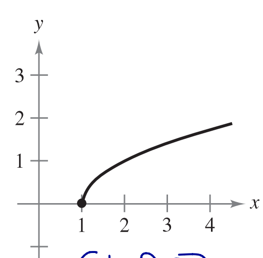
(a) $(-\infty, 4) \cup (4, \infty)$



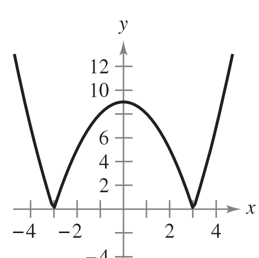
(b) $(-\infty, -4) \cup (-4, \infty)$



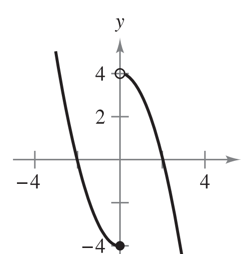
(c) $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$



(d) $(1, \infty)$



(e) $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$



(f) $(-\infty, 0) \cup (0, \infty)$

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

If a function is not continuous at $x = c$, it is also not differentiable at $x = c$.

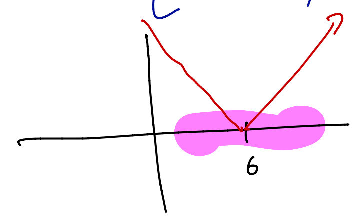
Ex.7 Use the alternative form of the derivative of $f(x) = |x-6|$ to find the derivative at $c = 6$.

$$f'(6) = \lim_{x \to 6} \frac{f(x) - f(6)}{x - 6}$$

$$= \lim_{x \to 6} \frac{|x-6| - |6-6|}{x-6}$$

$$= \lim_{x \to 6} \frac{|x-6|}{x-6} = \underline{DNE}$$

$$|x-6| = \begin{cases} +(x-6), & x \geq 6 \\ -(x-6), & x < 6 \end{cases}$$



Issue

$$\lim_{x \to 6^-} \frac{-(x-6)}{x-6} \qquad \lim_{x \to 6^+} \frac{+(x-6)}{x-6}$$

$$= \lim_{x \to 6^-} (-1) \qquad = \lim_{x \to 6^+} (+1)$$

$$= -1 \neq +1$$

Ex.8 Use the alternative form of the derivative of $f(x) = (x+3)^{\frac{1}{3}}$ to find the derivative at $c = -3$.

$$f'(-3) = \lim_{x \to -3} \frac{f(x) - f(-3)}{x - (-3)}$$

$$= \lim_{x \to -3} \frac{(x+3)^{1/3} - (-3+3)^{1/3}}{x+3}$$

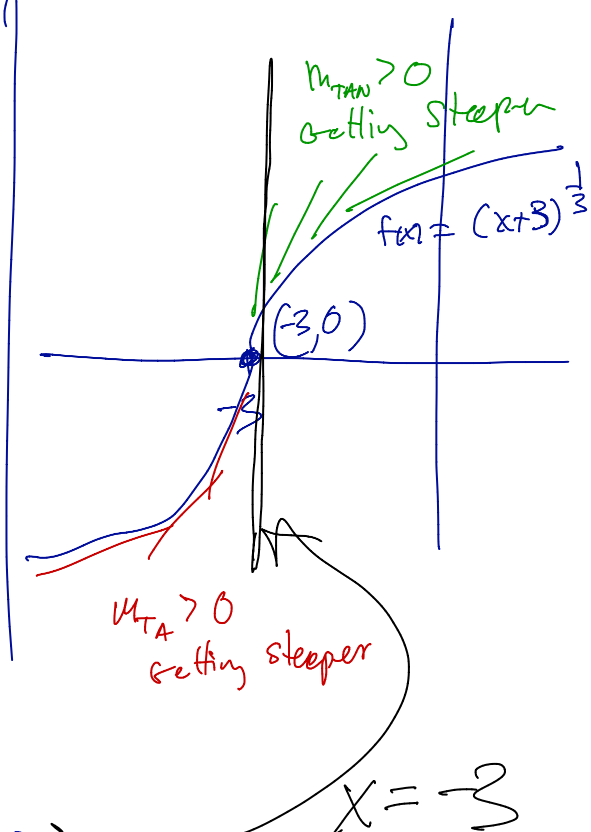
$$= \lim_{x \to -3} \frac{(x+3)^{1/3}}{(x+3)^{\frac{3}{3}}}$$

$$= \lim_{x \to -3} \frac{1}{(x+3)^{2/3}}$$

$= \underline{D}N\underline{E}.$

$= +\infty$

Vertical tangent line



THEOREM 2.1 Differentiability Implies Continuity

If f is differentiable at $x = c$, then f is continuous at $x = c$.

PROOF You can prove that f is continuous at $x = c$ by showing that $f(x)$ approaches $f(c)$ as $x \to c$. To do this, use the differentiability of f at $x = c$ and consider the following limit.

$$\begin{aligned}
\lim_{x\to c} [f(x) - f(c)] &= \lim_{x\to c} \left[(x - c)\left(\frac{f(x) - f(c)}{x - c}\right)\right] \\
&= \left[\lim_{x\to c} (x - c)\right]\left[\lim_{x\to c} \frac{f(x) - f(c)}{x - c}\right] \\
&= (0)[f'(c)] \\
&= 0
\end{aligned}$$

Because the difference $f(x) - f(c)$ approaches zero as $x \to c$, you can conclude that $\lim_{x\to c} f(x) = f(c)$. So, f is continuous at $x = c$. ■

The following statements summarize the relationship between continuity and differentiability.

1. If a function is differentiable at $x = c$, then it is continuous at $x = c$. So, differentiability implies continuity.
2. It is possible for a function to be continuous at $x = c$ and not be differentiable at $x = c$. So, continuity does not imply differentiability (see Example 6).

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$$\begin{aligned}\lim_{x\to c}[f(x) - f(c)] &= \lim_{x\to c}\left[(x - c)\left(\frac{f(x) - f(c)}{x - c}\right)\right]\\ &= \left[\lim_{x\to c}(x - c)\right]\left[\lim_{x\to c}\frac{f(x) - f(c)}{x - c}\right]\\ &= (0)[f'(c)]\\ &= 0\end{aligned}$$

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